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A Remark on the Farthest Point Problem

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1. INTRODUCTION

Let $(X, \|\cdot\|)$ be a normed linear space and $\phi \neq K \subset X$ be a bounded set. Let us define the mapping $Q_K\colon X \to 2^K$ by

$$Q_K(x) = \{y \in K;\ \sup_{k \in K} \| x - k \| = \| x - y \|\}.$$

We call K a uniquely remotal (U.R.) set, if for all $x \in X$ $Q_K(x)$ consists of exactly one element.

The following question seems to be unsolved: Are all (U.R.) sets singletons?

There are many cases in which this question can be answered affirmatively, such as for all finite-dimensional spaces [1]; for compact sets K [1]; for norm-continuous Q_K [2]; and for the Banach spaces c_0 and c [3]. In this paper we prove that for every normed linear space $(X, \|\cdot\|)$ there exists a $|\cdot|$ norm on X, equivalent to $\|\cdot\|$, such that all U.R. sets in $(X, |\cdot|)$ are singletons. The problem for the original norm remains unsolved.

2. RESULTS

THEOREM 1. *Let* $(X, |\cdot|)$ *and* $(Y, |\cdot|)$ *be arbitrary real normed linear spaces with* $\dim X > 0$, $\dim Y > 0$, *and let* $Z = X \times Y$ *have the norm*

$$\|(x, y)\| = \max\{| x |, | y |\}.$$

Then all U.R. *sets in* $(Z, \|\cdot\|)$ *are singletons.*

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LEMMA. *Let* $p_1, p_2 \in Z$, $p_i = (x_i, y_i)$ $(i = 1, 2)$; $Q_K(p_j) = Z_j = (a_j, b_j)$ $(j = 1, 2)$,

$$C_1 = \| p_1 - Q_K(p_1) \| = | X_1 - a_1 | \tag{1}$$

and

$$C_2 = \| p_2 - Q_K(p_2) \| = | y_2 - b_2 |. \tag{2}$$

Then $a_1 = a_2$, $b_1 = b_2$.

Proof of the Lemma. Let us suppose that $C_1 \geqslant C_2$, and set $\lambda = C_1/C_2$. Introducing the element

$$p_3 = \lambda p_2 + (1 - \lambda) Z_2,$$

we have

$$\| p_3 - Z_2 \| = \lambda C_2 = C_1 \tag{3}$$

and

$$\| p_3 - p_2 \| = (\lambda - 1) C_2 = C_1 - C_2.$$

Using (2) and the definition of Q_K,

$$\| p_3 - Q_K(p_3) \| \leqslant \| p_3 - p_2 \| + \| p_2 - Q_K(p_2) \| = \| p_3 - p_2 \| + C_2 = C_1.$$

By (3), this implies $Q_K(p_3) = Z_2$. By (2) and the definition $p_3 = (x_3, y_3)$,

$$| y_3 - b_2 | = C_1.$$

Set

$$p_4 = (X_1, y_3).$$

For arbitrary $Z \in K$, by (1) and (2),

$$\| p_4 - Z \| \leqslant \max(| X_1 - a_1 |, | y_3 - b_2 |) = C_1.$$

At the same time

$$\| p_4 - z_1 \| \geqslant | X_1 - a_1 | = C_1,$$

and

$$\| p_4 - z_2 \| \geqslant | y_3 - b_2 | = C_1.$$

So we have $Q_K(p_4) = z_1$ and $Q_K(p_4) = z_2$.

As $Q_K(p_4)$ is a singleton $z_1 = z_2$, and the lemma is proved.

Proof of Theorem 1. Suppose K is a U.R. set in $(Z, \| \cdot \|)$ which is not a singleton. First we show that there exist points p_1, p_2 in Z and points z_1, z_2

in K such that $Q_K(p_1) = z_1 \neq z_2 = Q_K(p_2)$. Otherwise we would have $Q_K(z) = z$ for all $z \in K$, contradicting our assumption that K is not a singleton.

Using our lemma we can assume, without loss of generality, that

$$\| p_i - z_i \| = | X_i - a_i | \qquad (i = 1, 2).$$

If there were a point $p = (X, y)$ in Z with $Q_K(p) = (a, b)$ and

$$\| p - Q_K(p) \| = | y - b |,$$

then, by the lemma, $z_1 = Q_K(p) = z_2$, contradicting the above. So we have

$$\| p - Q_K(p) \| = | X - a | \tag{4}$$

whenever $p = (X, y)$, $Q_K(p) = (a, b)$. But taking $p = (0, y)$ with $| y | \geqslant 3 \sup_{z \in K} \| z \|$, we have

$$\| p - Q_K(p) \| \geqslant 2 \sup_{z \in K} \| z \|$$

and

$$| 0 - a | \leqslant \sup_{z \in K} \| z \|$$

for all $(a, b) \in K$, contradicting (4). This proves Theorem 1.

THEOREM 2. *Let $(X, \| \cdot \|)$ be a real normed linear space. Then there exists a norm $| \cdot |$ on X, equivalent to $\| \cdot \|$, such that all* U.R. *sets in $(X, | \cdot |)$ are singletons.*

Proof. If $\dim X = 1$, our theorem is trivial.

If $\dim X \geqslant 2$, let $f \in (X, \| \cdot \|)^*$, $f \neq 0$. Then there exists $x^* \in X$ with $f(x^*) \neq 0$.

Set

$$M = \left\{ x - \frac{f(x)}{f(x^*)} x^*;\ x \in X \right\},$$

$$N = \{cx;\ c \in \mathbb{R}\}.$$

Then M, N are closed linear subspaces of X, and

$$M \oplus N = X.$$

We have

$$\|x\| \leqslant \left\| x - \frac{f(x)}{f(x^*)} x^* \right\| + \left\| \frac{f(x)}{f(x^*)} x^* \right\|$$

$$\leqslant 2 \max \left\{ \left\| x - \frac{f(x)}{f(x^*)} x^* \right\|, \left\| \frac{f(x)}{f(x^*)} x^* \right\| \right\}$$

and

$$\left\| \frac{f(x)}{f(x^*)} x^* \right\| \leqslant \|x^*\| \cdot \frac{\|f\|}{|f(x^*)|} \cdot \|x\|.$$

These inequalities show that

$$|x| = \max \left\{ \left\| x - \frac{f(x)}{f(x^*)} x^* \right\|, \left\| \frac{f(x)}{f(x^*)} x^* \right\| \right\} = \max\{\|P_M(x)\|, \|P_N(x)\|\}$$

is a norm, equivalent to $\|\cdot\|$. Here $P_M(P_N)$ denotes the projection operator P_M: $X \to M$, with ker $P_M = N$ (P_N: $X \to N$, with ker $P_N = M$). So, $(X, |\cdot|)$ can be regarded as the direct sum of $(M, \|\cdot\|)$ and $(N, \|\cdot\|)$.

By Theorem 1, all U.R. sets in $(X, |\cdot|)$ are singletons.

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